Operators on superspaces and generalizations of the Gelfand–Kolmogorov theorem

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Abstract. Gelfand and Kolmogorov in 1939 proved that a compact Hausdorff topological space X can be canonically embedded into the infinite-dimensional vector space $C(X)^*$, the dual space of the algebra of continuous functions $C(X)$, as an "algebraic variety", specified by an infinite system of quadratic equations.

Buchstaber and Rees have recently extended this to all symmetric powers $\mathrm{Sym}^n(X)$ using their notion of the Frobenius n-homomorphisms.

We give a simplification and a further extension of this theory, which is based, rather unexpectedly, on results from super linear algebra.

Keywords: Berezinian, superdeterminant, Gelfand–Kolmogorov theorem, Frobenius higher characters, n-homomorphisms, $p|q$-homomorphisms, symmetric powers, generalized symmetric powers, characteristic function of a linear map
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INTRODUCTION

In 1939 Gelfand and Kolmogorov proved [1] that any compact Hausdorff topological space X is canonically embedded into the infinite-dimensional vector space $C(X)^*$, the dual space of the algebra of continuous functions $C(X)$, as an "algebraic variety" specified by the infinite system of quadratic equations $\mathbf{f}(1) = 1$ and $\mathbf{f}(a^2) = \mathbf{f}(a)^2$ for linear functionals $\mathbf{f} \in C(X)^*$ indexed by elements $a \in C(X)$. Recently Buchstaber and Rees have suggested a generalization of the Gelfand–Kolmogorov theorem based on their notion of an n-homomorphism or 'Frobenius n-homomorphism'. They have showed that in fact all symmetric powers $\mathrm{Sym}^n(X)$ of the topological space X are canonically embedded into $C(X)^*$. To this end, the quadratic equations $\mathbf{f}(1) = 1$ and $\mathbf{f}(a^2) = \mathbf{f}(a)^2$, specifying the algebra homomorphisms, have to be replaced by more complicated algebraic equations. See [3, 4] and references therein. We have managed to find a different approach and a further generalization for this theory [6], which is motivated, rather unexpectedly, by ideas coming from considering linear operators acting on a superspace [5].

In the topic of this paper we see an interaction of ideas coming from different sources, some classical, and some quite new. They are: Frobenius's higher group characters; the Gelfand–Kolmogorov theorem; supergeometry and linear algebra on superspace (Berezin); multi-valued groups and the corresponding analog of Hopf algebras (Buchstaber and Rees). The latter lead to [2, 3, 4]. The study of linear operators on superspace lead to [5].

The main question that we shall discuss may be stated as follows. Consider a linear map between algebras A and B:

$$\mathbf{f}\colon\, A \to B\,.$$

What can be said about such a map? What are 'good classes' of maps of algebras? Our algebras are associative, with a unit, and commutative. (This can be slightly relaxed.) We consider algebras over $\mathbb{R}$ or $\mathbb{C}$.

Suppose $\mathbf{f}$ is an algebra homomorphism. The algebra homomorphisms have a clear geometrical meaning. Among all statements elaborating such an 'algebraic–functional duality', let us quote the following:

Theorem (Gelfand–Kolmogorov, 1939)**.** *Let $C(X)$ be the algebra of continuous functions on a compact Hausdorff topological space X. Then there is a one-to-one correspondence between the algebra homomorphisms $C(X) \to \mathbb{R}$ and the points of X. (All homomorphisms are the evaluation homomorphisms at $x \in X$).*

Here the algebra $A = C(X)$ is considered purely algebraically, without a topology. This theorem is less known than its analog where A is considered as a normed ring and homomorphisms are assumed to be continuous.

Since the homomorphism condition $\mathbf{f}(ab) = \mathbf{f}(a)\mathbf{f}(b)$ can be re-written, by using polarization, as $\mathbf{f}(a^2) = \mathbf{f}(a)^2$ for all $a \in A$, we arrive at the system of quadratic equations in the space A^* mentioned above. These equations describe the image of the embedding of X into A^*. Such an interpretation has been recently emphasized by Buchstaber and Rees, who gave an extension to all symmetric powers $\operatorname{Sym}^n(X)$.

In the main text below we explain a new idea allowing to obtain the statement of Buchstaber and Rees very simply. Moreover, following this path we obtain a further generalization. The main idea comes from our recent work on Berezinians and exterior powers [5]. In [6] one can find a more formal exposition. (The e-print version of [6] contains an appendix with details missing in the journal version.)

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A GENERALIZATION OF RING HOMOMORPHISMS

Motivated by their work on multi-valued groups, namely, by the properties of the algebras of functions on such generalization of groups, Buchstaber and Rees suggested the notion of n-homomorphisms of algebras, where $n = 1, 2, 3, \ldots$ Here 1-homomorphisms are ordinary algebra homomorphisms.

Recall the following construction, which can be traced back to Frobenius. For a given linear map $\mathbf{f}\colon\ A \to B$, define maps $\Phi_n\colon\ A \times \ldots \times A \to B$ by induction: $\Phi_1(a) = \mathbf{f}(a)$ and

$$\Phi_{k+1}(a_1, \ldots, a_{k+1}) = f(a_1)\Phi_k(a_2, \ldots, a_{k+1}) \\ - \Phi_k(a_1a_2, \ldots, a_{k+1}) - \ldots - \Phi_k(a_2, \ldots, a_1a_{k+1})\,.$$

In Frobenius's original work this was applied to a character of a linear representation of a finite group, producing the so-called 'Frobenius higher characters'. Although the definition is not manifestly symmetric, one can easily show by induction that the multilinear functions Φ_n are symmetric in their arguments. It follows that it is sufficient to consider

them on the diagonal.

Definition 1. An *n-homomorphism* $\mathbf{f}\colon A \to B$ is a linear map such that $\mathbf{f}(1) = n$ and $\Phi_{n+1} = 0$.

We shall say more about properties of n-homomorphisms in the next sections.

The main algebraic result of Buchstaber and Rees is the following.

Theorem (Buchstaber–Rees, 2002)**.** *There is a one-to-one correspondence between the n-homomorphisms $A \to B$ and the algebra homomorphisms $S^nA \to B$.*

Here $S^nA \subset A^{\otimes n}$ is the symmetric power of A considered as a subalgebra of the tensor power $A^{\otimes n}$. Geometrically this statement gives a canonical embedding of the symmetric power $\mathrm{Sym}^n(X) = X^n/S_n$ of a topological space X into $C(X)^*$ by a system of algebraic equations of higher order.

Example. Let $n = 2$. The embedding $\mathrm{Sym}^2(X) \to C(X)^*$ is given by the formulas

$$[x_1, x_2] \mapsto \mathbf{f} = \mathrm{ev}_{[x_1,x_2]} \quad \text{where} \quad \mathrm{ev}_{[x_1,x_2]}(a) = a(x_1) + a(x_2)\,.$$

The equations for a linear functional $\mathbf{f}\colon C(X) \to \mathbb{R}$ are

$$\mathbf{f}(1) = 2 \quad \text{and} \quad \begin{vmatrix} \mathbf{f}(a) & 1 & 0 \\ \mathbf{f}(a^2) & \mathbf{f}(a) & 2 \\ \mathbf{f}(a^3) & \mathbf{f}(a^2) & \mathbf{f}(a) \end{vmatrix} = 0 \quad \text{for all } a \in C(X)\,.$$

(The last equation is nothing but $\Phi_3 = 0$.)

Thus, Buchstaber and Rees introduced a class of maps of algebras generalizing homomorphisms and discovered their beautiful geometric properties. Buchstaber and Rees's original proofs are quite hard. See [3], where earlier works are summarized.

Now we shall explain how to obtain their results very quickly and how they can be extended. To do so we need a digression.

DIGRESSION: BEREZINIANS AND EXTERIOR POWERS

Recall the following definition. For an even invertible $p|q \times p|q$ matrix, $A = \begin{pmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{pmatrix}$, the *Berezinian* or *superdeterminant* is defined by

$$\mathrm{Ber}\, A = \frac{\det\left(A_{00} - A_{01}A_{11}^{-1}A_{10}\right)}{\det A_{11}}\,.$$

It is related with the *supertrace* $\mathrm{str}\, A = \mathrm{tr}\, A_{00} - \mathrm{tr}\, A_{11}$ by Liouville's relation

$$e^{\mathrm{str}\, A} = \mathrm{Ber}\, e^A\,.$$

In the ordinary case $q = 0$, $\mathrm{Ber} = \det$ and it is given by the action on the top exterior power of a vector space. In the super case, there is no such thing as the 'top exterior

power': the sequence $\Lambda^k(V)$ is infinite to the right. At the first glance there is no relation between Ber and exterior powers. However, the following was recently discovered.

Theorem ([5], 2003)**.** *If* $\dim V = p|q$*, then the following holds.*

(1) *The exterior powers* $\Lambda^k(V)$ *satisfy recurrence relations with* $q+1$ *terms in an appropriate Grothendieck ring for* $k \geqslant p-q+1$*. For any linear operator* A *on* V *there are 'universal recurrence relations' for the traces* $\operatorname{str}\Lambda^k(A)$*. This can be expressed by the equations*

$$\begin{vmatrix} c_k & \dots & c_{k+q} \\ \dots & \dots & \dots \\ c_{k+q} & \dots & c_{k+2q} \end{vmatrix} = 0$$

for $k \geqslant p-q+1$*. Here* c_k *are either* $\operatorname{str}\Lambda^k(A)$ *or* $\Lambda^k V$*.*

(2) *The Berezinian of a linear operator can be expressed as a ratio of polynomial invariants:*

$$\operatorname{Ber} A = \frac{\begin{vmatrix} c_{p-q} & \dots & c_p \\ \dots & \dots & \dots \\ c_p & \dots & c_{p+q} \end{vmatrix}}{\begin{vmatrix} c_{p-q+2} & \dots & c_{p+1} \\ \dots & \dots & \dots \\ c_{p+1} & \dots & c_{p+q} \end{vmatrix}} = \frac{|c_{p-q}\dots c_p|_{q+1}}{|c_{p-q+2}\dots c_{p+1}|_q},$$

where $c_k = \operatorname{str}\Lambda^k(A)$.

(The determinants involved in formulas above are the so-called Hankel determinants, which are minors of the infinite 'Hankel matrix' with the entries $c_{ij} = c_{i+j}$ corresponding to an infinite sequence c_k.)

The crucial tool for obtaining these and other results in [5] is the rational *characteristic function* of a linear operator

$$R_A(z) := \operatorname{Ber}(1+zA),$$

for which we consider expansions at zero and at infinity. As we shall see, this provides a new approach to the Gelfand–Kolmogorov–Buchstaber–Rees theory.

CHARACTERISTIC FUNCTION FOR A MAP OF ALGEBRAS

Let A and B be commutative associative algebras with unit. Consider an arbitrary linear map $\mathbf{f}\colon A \to B$. Mimicking constructions above, let us introduce the *characteristic function* for $\mathbf{f}$ as

$$R(\mathbf{f},a,z) := e^{\mathbf{f}\ln(1+az)},$$

where $a \in A$ and z is a formal parameter. Initially $R(\mathbf{f},a,z)$ is just a formal power series.

Example. Let $\mathbf{f}(a) = \operatorname{str}\rho(a)$ for a matrix representation ρ of the algebra A. Then $R(\mathbf{f},a,z) = \operatorname{Ber}(1+\rho(a)z) = R_{\rho(a)}(z)$, the characteristic function of the operator $\rho(a)$.

Let us turn to general maps of algebras $\mathbf{f}$.

Example. If $\mathbf{f}$ is an algebra homomorphism, then $R(\mathbf{f},a,z) = 1+\mathbf{f}(a)z$, i.e., a linear polynomial.

We see that algebraic properties of the map $\mathbf{f}$ are reflected in functional properties of $R(\mathbf{f},a,z)$ w.r.t. the variable z. What if $R(\mathbf{f},a,z)$ is a polynomial of degree n? We shall show in the next section that this corresponds to the n-homomorphisms of Buchstaber and Rees.

First let us discuss the general properties of $R(\mathbf{f},a,z)$. They are as follows (see [6]).

$R(\mathbf{f},a,z)$ satisfies the exponential property $R(\mathbf{f}+\mathbf{g},a,z) = R(\mathbf{f},a,z)R(\mathbf{g},a,z)$.

$R(\mathbf{f},a,z)$ has the explicit power expansion at zero

$$R(\mathbf{f},a,z) = 1+\psi_1(\mathbf{f},a)z+\psi_2(\mathbf{f},a)z^2+\dots$$

where $\psi_k(\mathbf{f},a) = P_k(s_1,\dots,s_k)$ with $s_k = \mathbf{f}(a^k)$ and P_k being the classical Newton polynomials giving expression of elementary symmetric functions via sums of powers:

$$P_k(s_1,\dots,s_k) = \frac{1}{k!}\begin{vmatrix} s_1 & 1 & 0 & \dots & 0 \\ s_2 & s_1 & 2 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ s_{k-1} & s_{k-2} & s_{k-3} & \dots & k-1 \\ s_k & s_{k-1} & s_{k-2} & \dots & s_1 \end{vmatrix}.$$

By induction one can check that $\Phi_k(a,\dots,a) = k!\,\psi_k(\mathbf{f},a)$, for the terms of the Frobenius recursion restricted to the diagonal.

Suppose now that $R(\mathbf{f},a,z)$ extends to a genuine function of z regarded, say, as a complex variable. Consider its behaviour at infinity. By a formal transformation one can see that $R(\mathbf{f},a,z) = z^{\mathbf{f}(1)}e^{\mathbf{f}\ln a}e^{\mathbf{f}\ln(1+a^{-1}z^{-1})}$. In particular, for $a=1$ we have $R(\mathbf{f},1,z) = (1+z)^{\mathbf{f}(1)}$. Assuming that $R(\mathbf{f},a,z)$ has no essential singularity we get that $\mathbf{f}(1) = \chi \in \mathbb{Z}$ is an integer, which is the order of the pole at infinity. Hence we have the expansion $R(\mathbf{f},a,z) = \sum_{k\leqslant\chi}\psi_k^*(\mathbf{f},a)z^k$ at infinity, where $\psi_k^*(\mathbf{f},a) := e^{\mathbf{f}\ln a}\psi_{\chi-k}(\mathbf{f},a^{-1})$. Denote the leading term of the expansion

$$\operatorname{ber}(\mathbf{f},a) := e^{\mathbf{f}\ln a}$$

and call it, the $\mathbf{f}$-*Berezinian* of $a\in A$.

One can immediately see that $\mathbf{f}$-Berezinian is multiplicative:

$$\operatorname{ber}(\mathbf{f},a_1a_2) = \operatorname{ber}(\mathbf{f},a_1)\operatorname{ber}(\mathbf{f},a_2).$$

APPLICATION: BUCHSTABER–REES THEORY

Suppose that the characteristic function $R(\mathbf{f},a,z)$ is a polynomial for all a. In particular it follows that $\chi = \mathbf{f}(1)$ must be positive; denote it $n\in\mathbb{N}$. Hence it is the degree of $R(\mathbf{f},a,z)$ for all a. So $\psi_k(\mathbf{f},a) = 0$ for all $k\geqslant n+1$ and all $a\in A$. This is equivalent to the equations $\mathbf{f}(1) = n\in\mathbb{N}$ and $\Phi_{n+1}(\mathbf{f},a_1,\dots,a_{n+1}) = 0$ for all a_i, which is precisely the definition of an n-homomorphism according to Buchstaber and Rees [3].

Various properties of n-homomorphisms immediately follow from this description. For example, the exponential property of the characteristic function implies that the sum of an n-homomorphism and an m-homomorphism is an $(n+m)$-homomorphism. Similarly one can deduce that the composition of n-homomorphism and an m-homomorphism is an nm-homomorphism. (These results were originally obtained much harder, see [3, 4].)

The main theorem of Buchstaber and Rees can be easily obtained as follows.

Since $\mathbf{f}$-Berezinian is multiplicative, and for n-homomorphisms $\operatorname{ber}(\mathbf{f},a) = \psi_n(\mathbf{f},a)$, the function $\psi_n(\mathbf{f},a)$ is multiplicative in a. Therefore its polarization $\frac{1}{n!}\Phi_n(\mathbf{f},a_1,\ldots,a_n)$ yields an algebra homomorphism $S^nA \to B$. Thus a one-to-one correspondence between the n-homomorphisms $A \to B$ and the algebra homomorphisms $S^nA \to B$ is established. The transparency of this proof illustrates the power of our approach. (The multiplicativity of $\frac{1}{n!}\Phi_n(\mathbf{f},a,\ldots,a)$ was the hardest part of the original proof [2]; it was obtained there by using a non-trivial combinatorics.)

FURTHER EXTENSION: GENERALIZED SYMMETRIC POWERS

Suppose the characteristic function $R(\mathbf{f},a,z)$ is not a polynomial, but a rational function. We arrive at a further generalization of ring homomorphisms.

Definition 2. We call a linear map $\mathbf{f}\colon A \to B$ a *$p|q$-homomorphism* if $R(\mathbf{f},a,z)$ can be written as the ratio of polynomials of degrees p and q.

We have $\chi = \mathbf{f}(1) = p - q$ for $p|q$-homomorphisms.

Examples. The negative $-\mathbf{f}$ of a ring homomorphism $\mathbf{f}$ is a $0|1$-homomorphism. The difference $\mathbf{f}_{(p)} - \mathbf{f}_{(q)}$ of a p-homomorphism $\mathbf{f}_{(p)}$ and a q-homomorphism $\mathbf{f}_{(q)}$ is a $p|q$-homomorphism. In particular, a linear combination of algebra homomorphisms of the form $\sum n_\alpha \mathbf{f}_\alpha$ where $n_\alpha \in \mathbb{Z}$ is a $p|q$-homomorphism with $\chi = \sum n_\alpha$, $p = \sum\limits_{n_\alpha > 0} n_\alpha$, and $q = -\sum\limits_{n_\alpha < 0} n_\alpha$. It all follows from the exponential property of the characteristic function.

By using formulas from [5], the condition that $\mathbf{f}\colon A \to B$ is a $p|q$-homomorphism can be expressed by the equations

$$\mathbf{f}(1) = p - q \quad \text{and} \quad \begin{vmatrix} \psi_k(\mathbf{f},a) & \ldots & \psi_{k+q}(\mathbf{f},a) \\ \ldots & \ldots & \ldots \\ \psi_{k+q}(\mathbf{f},a) & \ldots & \psi_{k+2q}(\mathbf{f},a) \end{vmatrix} = 0 \tag{1}$$

(the Hankel determinant), for all $k \geqslant p - q + 1$.

What is the geometrical meaning of this notion?

Consider a topological space X. We define its *$p|q$-th symmetric power* $\operatorname{Sym}^{p|q}(X)$ as the identification space of X^{p+q} with respect to the action of $S_p \times S_q$ and the relations

$$(x_1,\ldots,x_{p-1},y,x_{p+1}\ldots,x_{p+q-1},y) \sim (x_1,\ldots,x_{p-1},z,x_{p+1}\ldots,x_{p+q-1},z)\,.$$

The algebraic analog of $\operatorname{Sym}^{p|q}(X)$ is the *$p|q$-th symmetric power* $S^{p|q}A$ of a commutative associative algebra with unit A. We define $S^{p|q}A$ as the subalgebra

$\mu^{-1}\left(S^{p-1}A\otimes S^{q-1}A\right)$ in $S^pA\otimes S^qA$ where $\mu\colon\ S^pA\otimes S^qA\to S^{p-1}A\otimes S^{q-1}A\otimes A$ is the multiplication of the last arguments.

Example. For $A=\mathbb{C}[x]$, the algebra $S^{p|q}A$ is the algebra of all polynomial invariants of $p|q$ by $p|q$ matrices. (This is a non-trivial statement, see in [5].)

There is a relation between algebra homomorphisms $S^{p|q}A\to B$ and $p|q$-homomorphisms $A\to B$. To each homomorphism $S^{p|q}A\to B$ canonically corresponds a $p|q$-homomorphism $A\to B$.

Example. An element $\mathbf{x}=[x_1,\ldots,x_{p+q}]\in\mathrm{Sym}^{p|q}(X)$ defines the $p|q$-homomorphism $\mathrm{ev}_{\mathbf{x}}\colon\ C(X)\to\mathbb{R}$:

$$a\mapsto a(x_1)+\ldots+a(x_p)-\ldots-a(x_{p+q})\,.$$

This gives a natural map $\mathrm{Sym}^{p|q}(X)\to A^*$, where $A=C(X)$, which generalizes the Gelfand–Kolmogorov and Buchstaber–Rees maps (in fact, an embedding). The image of $\mathrm{Sym}^{p|q}(X)$ in A^* satisfies equations (1) where $\mathbf{f}=\mathrm{ev}_{\mathbf{x}}$. It is a system of polynomial equations for 'coordinates' of a linear map $\mathbf{f}\in A^*$.

A conjectured statement is that the solutions of equations (1) give precisely the image of $\mathrm{Sym}^{p|q}(X)$. This would be an exact analog of the Gelfand–Kolmogorov and Buchstaber–Rees theorems. The corresponding algebraic statement would be a one-to-one correspondence between the $p|q$-homomorphisms $A\to B$ and the algebra homomorphisms $S^{p|q}A\to B$.

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REFERENCES

1. I. M. Gel'fand and A. N. Kolmogorov, *Dokl. Akad. Nauk SSSR* **22**, 11–15 (1939).
2. V. M. Buchstaber and E. G. Rees, *Selecta Math. (N.S.)* **8**(4), 523–535 (2002).
3. V. M. Buchstaber and E. G. Rees, *Russian Math. Surveys* **59**(1), 125–145 (2004).
4. V. M. Buchstaber and E. G. Rees, *Math. Proc. Cambr. Phil. Soc.*, to appear (2007), `arXiv:math.RA/0608120`.
5. H. M. Khudaverdian and Th. Th. Voronov, *Lett. Math. Phys.* **74**(2), 201–228 (2005).
6. H. M. Khudaverdian and Th. Th. Voronov, *Russian Math. Surveys* **62** (3), 209–210 (2007), `arXiv:math.RA/0612072`.